

SUBMONOIDS OF THE FORMAL POWER SERIES

EDGAR ENOCHS, OVERTOUN JENDA, AND FURUZAN OZBEK

ABSTRACT. Formal power series come up in several areas such as formal language theory, algebraic and enumerative combinatorics, semigroup theory, number theory etc. ([2], [4], [1], [6], [5]). This paper focuses on the subset $xR[[x]]$ consisting of formal power series with zero constant term. This subset forms a monoid with the composition operation on series. We classify the sets T of strictly positive integers for which the set of formal power series

$$R[[x^T]] = \left\{ \sum_{t \in T} a_t x^t \mid \text{where } a_t \in R \right\}$$

forms a monoid with composition as the operation. We prove that in order for $R[[x^T]]$ to be a monoid, T itself has to be a submonoid of (\mathbb{N}, \cdot) . Unfortunately, this condition is not enough to guarantee the desired result. But if a monoid is *strongly closed*, then we get the desired result. We also consider an analogous problem for power series in several variables.

1. INTRODUCTION

Let R be a ring with identity and $R[[x]]$ denote the ring of formal power series in the indeterminate x with coefficients in R . Throughout the paper we will denote the natural numbers with \mathbb{N} and the non-zero natural numbers with \mathbb{N}^* . Then $xR[[x]] = R[[x^{\mathbb{N}^*}]]$ will denote the series with zero constant term. The order of a formal power series $f(x) = \sum a_n x^n$ is the smallest power of x appearing in the infinite sum and denoted by $\omega(f(x))$ (i.e. $\omega(f(x)) = n$ if $a_n \neq 0$ and if $a_i = 0$ for all $0 \leq i < n$, and $\omega(0) = \infty$). Note that $R[[x]]$ forms a group with respect to addition but not with composition. In this paper, we focus on $xR[[x]]$ which forms a monoid with composition as the operation. Composition is defined as usual (i.e. $f, g \in xR[[x]]$, $f \circ g(x) = f(g(x))$). The identity of $xR[[x]]$ is x and its subset consisting of formal power series with $f'(0)$ a unit of R forms a group.

There are some obvious submonoids of $xR[[x]]$. For example if $k \geq 2$, the set

$$U = \{f(x) \in xR[[x]] \mid f(x) = a_1x + a_kx^k + a_{k+1}x^{k+1} + \dots\}$$

forms a submonoid. If $d \geq 1$, another useful submonoid is

$$V = \{f(x) \in xR[[x]] \mid f(x) = a_1x + a_2x^{1+d} + a_3x^{1+2d} + \dots\}$$

Note that in each case we can specify the submonoid by considering a subset $T \subset \mathbb{N}^*$ and then taking a series in x^t with $t \in T$. This set of series will be denoted $R[[x^T]]$. Our object is to find all $T \subset \mathbb{N}^*$ such that $R[[x^T]]$ is a submonoid of $R[[x^{\mathbb{N}^*}]]$. In order for $R[[x^T]]$ to be a submonoid of $R[[x^{\mathbb{N}^*}]]$ there are two necessary conditions. Since we want $x \in R[[x^T]]$, we must have $1 \in T$. Also if $s, t \in T$ then $x^s \circ x^t = x^{st}$ must be in $R[[x^T]]$, so we must have $st \in T$. That is T must be a submonoid of the multiplicative monoid \mathbb{N}^* .

Remark 1.1. *If $R[[x^T]]$ is a submonoid of $R[[x^{\mathbb{N}^*}]]$ then T is a submonoid of \mathbb{N}^* with respect to multiplication.*

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The above remark gives a necessary condition for our goal but it is not sufficient. To see this consider the multiplicative submonoid $T = \{1, 2, 4, 6, \dots\}$ of \mathbf{N}^* . Note that the corresponding series $R[[x^T]]$ is not a submonoid, since $x^2 \circ (x + x^2) \notin R[[x^T]]$. So we need a stronger condition on T to guarantee that $R[[x^T]]$ is a submonoid of $R[[x^{\mathbf{N}^*}]]$.

2. MAIN RESULT

In the previous section we concluded that T being a multiplicative submonoid is not sufficient for our goal. The following theorem gives us equivalent sufficient conditions on T for $R[[x^T]]$ to be a submonoid of $R[[x^{\mathbf{N}^*}]]$.

Theorem 2.1. *If $T \subset \mathbf{N}^*$, then the following are equivalent:*

- a): $R[[x^T]]$ is a submonoid of $R[[x^{\mathbf{N}^*}]]$ for any ring R .
- b): $1 \in T$ and if $s, t_1, t_2, \dots, t_k \in T$ then for any partition $s = s_1 + s_2 + \dots + s_k$ we have $s_1 t_1 + \dots + s_k t_k \in T$.
- c): $1 \in T$ and if $s, t \in T$ then $s + t - 1 \in T$.

Proof. (a \Rightarrow b) As we noted earlier, $x \in R[[x^T]]$ so we must have $1 \in T$.

Let $R = \mathbb{Z}$ and $s, t_1, t_2, \dots, t_k \in T$. Then $x^{t_1} + \dots + x^{t_k}, x^s$ are in $R[[x^T]]$ and by our assumption so is $x^s \circ (x^{t_1} + \dots + x^{t_k}) = (x^{t_1} + \dots + x^{t_k})^s$. When we expand $(x^{t_1} + \dots + x^{t_k})^s$ using multinomial theorem we see that all coefficients are non-negative integers. One such coefficient is $\binom{s}{s_1, \dots, s_k}$ where $s = s_1 + \dots + s_k$, i.e. the coefficient of $x^{s_1 t_1 + \dots + s_k t_k}$ in the expansion. So when we collect like terms the coefficient of $x^{s_1 t_1 + \dots + s_k t_k}$ is a strictly positive integer. That is, $s_1 t_1 + \dots + s_k t_k$ must be in T .

(b \Rightarrow a) Since $1 \in T$ we have $x \in R[[x^T]]$ for any R . So we only need to show closure of $R[[x^T]]$ for composition. Note that $R[[x^T]]$ is closed under addition, multiplication by $a \in R$ and taking limits, we will have closure for composition if we can show that $x^s \circ (a_1 x^{t_1} + \dots + a_k x^{t_k}) = (a_1 x^{t_1} + \dots + a_k x^{t_k})^s \in R[[x^T]]$ whenever $s, t_1, \dots, t_k \in T$ for some $k \geq 1$ and $a_1, \dots, a_k \in R$. Now it is easy to see that if we multiply $(a_1 x^{t_1} + \dots + a_k x^{t_k})^s$ out and collect like terms, the only possible powers of x with non-zero coefficient are those of the form $s_1 t_1 + \dots + s_k t_k$ for some partition $s = s_1 + \dots + s_k$ of s .

(b \Rightarrow c) By assumption $1 \in T$. Let s, t be in T . Then $t + s - 1 = 1 \cdot t + \overbrace{1 \cdot 1 + \dots + 1 \cdot 1}^{s-1}$. So we use the partition $s = 1 + \dots + 1$ and the fact that $t, 1, \dots, 1 \in T$ to get $t + s - 1$ is in T .

(c \Rightarrow b) Again we get $1 \in T$. Given $s, t_1, \dots, t_k \in T$ and any partition $s = s_1 + \dots + s_k$, we rewrite $s_1 t_1 + \dots + s_k t_k = (s_1 + \dots + s_k) + s_1(t_1 - 1) + \dots + s_k(t_k - 1) = s + s_1(t_1 - 1) + \dots + s_k(t_k - 1)$ and a repeated application of (c) s -many times gives the desired result. \square

Definition. A subset $T \subset \mathbf{N}^*$ satisfying the equivalent conditions of Theorem 2.1 is said to be a *strongly closed submonoid* of \mathbf{N}^* (or simply strongly closed).

3. PROPERTIES OF STRONGLY CLOSED SUBMONOIDS

In this section we investigate properties of strongly closed submonoids and the relationship between the strongly closed submonoids of \mathbf{N}^* and the submonoids of \mathbf{N} with respect to addition.

Proposition 3.1. *The subset $T \subset \mathbf{N}^*$ is strongly closed if and only if $S = \{t - 1 \mid t \in T\}$ is a submonoid of \mathbf{N} (with respect to addition).*

Proof. Let $T \subset \mathbf{N}^*$ be strongly closed. Since $1 \in T$ we have $1 - 1 = 0 \in S$. If $s_1 = t_1 - 1, s_2 = t_2 - 1$ are in S , where $t_1, t_2 \in T$, then notice that $\underbrace{t_1 + t_2 - 1}_{\in T} - 1 = t_1 - 1 + t_2 - 1 = s_1 + s_2 \in S$, that is S is a submonoid of \mathbf{N} .

Now let $S \subset \mathbf{N}$ be a submonoid with respect to addition and let $T = \{1 + s \mid s \in S\}$. Then since $0 \in S$ we have $1 + 0 = 1 \in T$. If $t_1 = 1 + s_1, t_2 = 1 + s_2 \in T$, where $s_1, s_2 \in S$, then $t_1 + t_2 - 1 = 1 + \underbrace{s_1 + s_2}_{\in S} \in T$. \square

By proposition 3.1, we conclude that there is a bijective correspondence between strongly closed submonoids of \mathbf{N}^* and submonoids of $(\mathbf{N}, +)$. This characterization leads us to the next result which shows that strongly closed submonoids are always finitely generated.

Definition. Let $X \subset \mathbf{N}^*$, then the strong submonoid generated by X is defined to be the smallest strongly closed submonoid containing X .

Proposition 3.2. *Let $X \subset \mathbf{N}^*$, then the strong submonoid generated by X has the form*

$$T = \{1 + \sum_{a \in X} u_a(a-1) \mid u_a \geq 0 \text{ and } u_a = 0 \text{ for all but finitely many } a's\}$$

Proof. First, let us prove that T is strongly closed. Clearly $1 \in T$ and $X \subset T$. Suppose $u = 1 + \sum_{a \in X} u_a(a-1)$ and $v = 1 + \sum_{a \in X} v_a(a-1)$ are in T , then $u+v-1 = 1 + \sum_{a \in X} (u_a+v_a)(a-1) \in T$ since only finitely many of u_a, v'_a 's are non-zero and $u+v-1$ is in the desired form.

Now we need to prove that T is the smallest strongly closed submonoid containing X . Let \tilde{T} be another strongly closed submonoid containing X . Notice that by successive applications of part (c) of Theorem 2.1 we get,

$$a_1, \dots, a_k \in \tilde{T} \Rightarrow a_1 + a_2 + \dots + a_k - (k-1) \in \tilde{T}$$

So we conclude that for any non-negative integer u_a and $a \in X$, $1 + \sum_{a \in X} u_a(a-1)$ is in \tilde{T} . \square

Corollary 3.3. *There is a bijective correspondence between the set of submonoids $S \subset \mathbf{N}$ (with respect to addition) and the strong submonoids $T \subset \mathbf{N}^*$. The correspondence is given by $T = 1+S$. Moreover, if T is generated by X then S is generated by the set $Y = X-1 = \{a-1 \mid a \in X\}$.*

Proof. The result follows from the proposition 3.1. \square

Corollary 3.4. *Every strongly closed submonoid $T \subset \mathbf{N}^*$ is finitely generated as a closed submonoid.*

Proof. The result follows from corollary 3.3 and the fact that every submonoid of $(\mathbf{N}, +)$ is finitely generated. Hence, by the bijective correspondence one can get finite generating sets for the corresponding strongly closed submonoids. \square

Surprisingly enough a strongly closed submonoid $T \subset \mathbf{N}^*$ turns out to be an infinitely generated monoid with respect to multiplication.

Proposition 3.5. *Let $T \subset \mathbf{N}^*$ be a strongly closed submonoid, then T is an infinitely generated monoid with multiplication as the operation.*

Proof. Assume that T is a strongly closed submonoid and let $a \in T$ be an element different from 1. So $a > 1$, then $a + k(a-1) \in T$ for any $k \geq 0$. But $\gcd(a, a-1) = 1$, and by Dirichlet's theorem (see [3]) there are infinitely many primes among $a + k(a-1)$. So any such prime must be in the set of generators of T as a monoid (w.r.t. multiplication). We conclude that there is no finite generating set of T w.r.t. multiplication. \square

4. GENERALIZATION TO n -VARIABLE CASE

In this section we consider our problem in higher dimension. Let $n \geq 2$ and consider the ring $R[[x_1, \dots, x_n]]$ of formal power series in x_1, x_2, \dots, x_n with coefficients in R . If we consider n -tuples $f = (f_1, \dots, f_n)$ where $f_i \in R[[x_1, \dots, x_n]]$ and we define composition by

$$(1) \quad (f_1, \dots, f_n) \circ (g_1, \dots, g_n) = (f_1(g_1, \dots, g_n), \dots, f_n(g_1, \dots, g_n))$$

then we get a monoid $R[[x_1, \dots, x_n]]^n$ with the identity (x_1, \dots, x_n) .

For any $u = (u_1, \dots, u_n) \in \mathbf{N}^n$, we denote $x^u = x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$. Our problem in this section will be to find the subsets $U \subset (\mathbf{N}^n)^* = \mathbf{N}^n - (0, \dots, 0)$ such that $f = (f_1, \dots, f_n)$ with $f_i \in R[[x^U]]$ (the formal power series in x^u with $u \in U$) form a submonoid of $R[[x^{(\mathbf{N}^n)^*}]]^n$. Since (x_1, \dots, x_n) is the identity with respect to multiplication in n -dimension, we want each x_i to be in $R[[x^U]]$ for all $i = 1, \dots, n$. Note that with the previous notation $x^{e_i} = x_i$ where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with only non-zero entry at the i -th slot, then each e_i must be in U for all $i = 1, \dots, n$.

For $u = (u_1, \dots, u_n)$ in \mathbf{N}^n , we define its norm as $|u| = u_1 + \dots + u_n$. Note that for $u, v \in \mathbf{N}^n$, $|u + v| = |u| + |v|$. Now we are ready to give the analogous result for the n -variable case.

Proposition 4.1. *Let $R[[x^U]]^n$ be a submonoid of $R[[x^{(\mathbf{N}^n)^*}]]^n$ with the composition operation defined as in (1). If $u \in U$ and $v \in (\mathbf{N}^n)^*$ such that $|v| = |u|$, then $v \in U$.*

Proof. First we conclude that $x_1, \dots, x_n, g = x_1 + \dots + x_n \in R[[x^U]]$. Let $u = (u_1, \dots, u_n) \in U$, then $f = x_1^{u_1} \dots x_n^{u_n} \in R[[x^U]]$ and since $R[[x^U]]^n$ is a submonoid of $R[[x^{(\mathbf{N}^n)^*}]]^n$ we conclude,

$$f \circ \overbrace{(g, \dots, g)}^{n\text{-tuple}} = f(g, \dots, g) = g^{u_1} \dots g^{u_n} = g^{u_1 + \dots + u_n} \in R[[x^U]]$$

So we need $g^{|u|} = (x_1 + \dots + x_n)^{|u|}$ to be in $R[[x^U]]$ for each R . In particular, if $R = \mathbb{Z}$, and $|u| = v_1 + \dots + v_n$ is another partition of $|u|$ then using the multinomial theorem on $(x_1 + \dots + x_n)^{|u|}$ we get the term $\binom{|u|}{v_1, \dots, v_n} x_1^{v_1} \dots x_n^{v_n}$. Since all the coefficients in this expansion of $(x_1 + \dots + x_n)^{|u|}$ are non-negative integers, we see that $v = (v_1, \dots, v_n)$ must be in U . \square

Furthermore, there is an intrinsic relationship between the set U of n -tuples and strongly closed submonoids as explained in details (theorem 4.3). But we need following lemma in order to prove this relationship.

Lemma 4.2. *Let T be a strongly closed submonoid of \mathbf{N}^* and $U = \{u \in (\mathbf{N}^n)^* \mid |u| \in T\}$. Then $f(g_1, \dots, g_n) \in R[[x^U]]$ for all $f, g_1, g_2, \dots, g_n \in R[[x^U]]$ if $(x^{u_1} + \dots + x^{u_m})^{|v|} \in \mathbb{Z}[[x^U]]$ for all $u_1, \dots, u_m, v \in U$.*

Proof. Let $f, g_1, g_2, \dots, g_n \in R[[x^U]]$, we will show that $f(g_1, \dots, g_n) \in R[[x^U]]$. It suffices to argue this holds for $f = x^v$ for some $v = (v_1, \dots, v_n) \in U$ as $R[[x^U]]$ is closed under addition, scalar multiplication and taking limits. So we want $g_1^{v_1} \dots g_n^{v_n} \in R[[x^U]]$. Since $R[[x^U]]$ is closed under limits, we may assume that g_i 's are polynomials in $R[[x^U]]$ and say a linear combination of x^{u_1}, \dots, x^{u_m} for some $u_1, \dots, u_m \in U$ with coefficients in R . Let $g_i = a_1^i x^{u_1} + \dots + a_m^i x^{u_m}$ for $i = 1, \dots, n$.

Note that if the coefficient of x^w in the product $g_1^{v_1} \dots g_n^{v_n}$ is zero, then w does not have to be in U . Now assume that the coefficient of x^w is non-zero. But then we claim that the coefficient of x^w in $(x^{u_1} + \dots + x^{u_m})^{|v|} \in \mathbb{Z}[[x]]$ is non-zero as well. This follows by the multinomial theorem because if the coefficient of x^w is non-zero in the product $g_1^{v_1} \dots g_n^{v_n}$ then there must be a partition of v say $|v| = t_1 + \dots + t_m$, so that $w = u_1 t_1 + \dots + u_m t_m$. But this means $(x^{u_1} + \dots + x^{u_m})^{|v|} = \binom{|v|}{t_1, \dots, t_m} x^w + \dots$ in $\mathbb{Z}[[x^U]]$ and that the coefficient of x^w is non-zero in $(x^{u_1} + \dots + x^{u_m})^{|v|}$. Hence $w \in U$. \square

Theorem 4.3. $R[[x^U]]^n$ is a submonoid of $R[[x^{(\mathbf{N}^n)^*}]]^n$ with the composition operation given as in (1) if and only if the set defined by $T = \{|u| \mid u \in U\}$ is a strongly closed submonoid of \mathbf{N}^* .

Proof. First, let us assume that $U \subset (\mathbf{N}^n)^*$ is such that $R[[x^U]]^n$ is a submonoid of $R[[x^{(\mathbf{N}^n)^*}]]^n$. Since $e_1 \in U$, we have $|e_1| = 1 \in T$. Now let $|u|, |v| \in T$ where $u, v \in U$. Since $(|u|, 0, \dots, 0)$ has the same absolute value as u , by proposition 4.1 we conclude that $(|u|, 0, \dots, 0) \in U$. Similarly $(|v|, 0, \dots, 0) \in U$. So $x^{e_1} + x^{(|u|, 0, \dots, 0)} = x_1 + x_1^{|u|} \in R[[x^U]]$. Substituting, $x_1 + x_1^{|u|}$ for x , in $x^{(|v|, 0, \dots, 0)} = x_1^{|v|}$ we get,

$$\begin{aligned} (x_1 + x_1^{|u|})^{|v|} &= x_1^{|v|} + |v|x_1^{|v|-1}x_1^{|u|} + \dots \\ &= x_1^{|v|} + |v|x_1^{|u|+|v|-1} + \dots \end{aligned}$$

Letting $R = \mathbb{Z}$ we see that we must have $(|u| + |v| - 1, 0, \dots, 0) \in U$. So its absolute value, i.e. $|u| + |v| - 1$, must be in T and T is strongly closed.

To show the converse, assume that $T \subset \mathbf{N}^*$ is a strongly closed submonoid. Let $U = \{u \in (\mathbf{N}^n)^* \mid |u| \in T\}$. Since $1 \in T$ and since $|e_i| = 1$ for $0 \leq i \leq n$, we have $e_1, \dots, e_n \in U$. So $x^{e_i} = x_i \in R[[x^U]]$ for $i = 1, \dots, n$.

By lemma 4.2, $f(g_1, \dots, g_n) \in R[[x^U]]$ for all $f, g_1, g_2, \dots, g_n \in R[[x^U]]$ if $(x^{u_1} + \dots + x^{u_m})^{|v|} \in \mathbb{Z}[[x^U]]$ for all $u_1, \dots, u_m, v \in U$. So we just need to prove that $(x^{u_1} + \dots + x^{u_m})^{|v|} \in \mathbb{Z}[[x^U]]$ whenever $u_1, \dots, u_m, v \in U$. Assume we are given $u_1, \dots, u_m, v \in U$ by multionomial theorem, it suffices to show that $s_1 u_1 + \dots + s_m u_m \in U$ for any partition $|v| = s_1 + \dots + s_m$. Notice that, this is the case if and only if $|s_1 u_1 + \dots + s_m u_m| = s_1 |u_1| + \dots + s_m |u_m| \in T$, but that holds as T is strongly closed. So we conclude that $f(g_1, \dots, g_n) \in R[[x^U]]$. \square

5. THE INVERTIBLE ELEMENTS OF $R[[x^T]]$

We turn our attention to the invertible elements of $R[[x]]$ with respect to composition. Throughout this section, invertibility will be understood with respect to composition. It is easy to see that a formal power series f is invertible if and only if its constant term is zero and $f'(0)$ is invertible in R . We show that for an arbitrary $T \subseteq \mathbf{N}^*$ the subset consisting of the invertible elements of $R[[x^T]]$ forms a subgroup if and only if T is strongly closed.

Theorem 5.1. Let $T \subseteq \mathbf{N}^*$. Then the subset $H \subset R[[x^T]]$ of invertible elements forms a group with composition as the operation for any ring R if and only if T is strongly closed.

Proof. Assume that the subset $H \subset R[[x^T]]$ of invertible elements forms a group. Then $x \in H$ implies $1 \in T$. So we only need to show that if $s, t \in T$ then $s + t - 1 \in T$. This is trivial if $s = 1$ or $t = 1$, so assume $s, t \geq 2$. Let $R = \mathbb{Z}$, then both $x + x^t, x + x^s \in \mathbb{Z}[[x^T]]$ are invertible and,

$$(x + x^s) \circ (x + x^t) = x + x^t + (x + x^t)^s \in H \subseteq \mathbb{Z}[[x^T]]$$

We have,

$$(x + x^t)^s = x^s + \binom{s}{1} x^{s-1+t} + \binom{s}{2} x^{s-2+2t} + \dots$$

Note that since $s, t \geq 2$ we have $s < s - 1 + t < s - 2 + 2t < \dots$. So the exponent $s + t - 1$ appears in $(x + x^t)^s$. It doesn't appear in $x + x^t$, so it appears in $x + x^t + (x + x^t)^s$, hence $s + t - 1 \in T$.

Now we assume that T is strongly closed. It is enough to show that if $f \in R[[x^T]]$ is invertible then f^{-1} is in $R[[x^T]]$ as well. Let $T = \{1 = t_1 < t_2 < t_3 < \dots\}$ and $f(x) = \sum_{t \in T} a_t x^t$ be invertible.

Say $f^{-1}(x) = \sum_{n=1}^{\infty} b_n x^n$, we will show inductively that if $b_n \neq 0$ then $n \in T$. For $n = 1$ there is

nothing to show since $1 \in T$. So suppose for each of b_1, \dots, b_{n-1} that either it is 0 or that when $b_k \neq 0$ then $k \in T$ for $1 \leq k \leq n-1$. We have $f(x) \circ f^{-1}(x) = x$ truncating we get,

$$f(x) \circ (b_1x + \dots + b_nx^n) \cong x \pmod{x^{n+1}}$$

That is,

$$a_1(b_1x + \dots + b_nx^n) + a_{t_2}(b_1x + \dots + b_nx^n)^{t_2} + \dots \cong x \pmod{x^{n+1}}$$

But note that $a_{t_2}(b_1x + \dots + b_nx^n)^{t_2} \cong a_{t_2}(b_1x + \dots + b_{n-1}x^{n-1})^{t_2} \pmod{x^n}$ since $n \geq 2$. Likewise for $a_{t_3}(b_1x + \dots + b_nx^n)^{t_3}$ etc. So we have,

$$a_1(b_1x + \dots + b_nx^n) + \underbrace{a_{t_2}(b_1x + \dots + b_{n-1}x^{n-1})^{t_2} + a_{t_3}(b_1x + \dots + b_{n-1}x^{n-1})^{t_3} + \dots}_* \cong x \pmod{x^{n+1}}$$

By induction hypothesis with $n \geq 2$, $(b_1x + \dots + b_{n-1}x^{n-1})^{t_2}$ and $(b_1x + \dots + b_{n-1}x^{n-1})^{t_3}$ are in $R[[x^T]]$. So $*$ $\in R[[x^T]]$.

Now investigating the coefficient of x^n in $a_1(b_1x + \dots + b_nx^n) + *$, we see that it is $a_1b_n +$ the coefficient of x_n in $*$. But this coefficient must be zero since,

$$a_1(b_1x + \dots + b_nx^n) + * \cong x \pmod{x^{n+1}}$$

and since $n \geq 2$.

So if $b_n \neq 0$, then $a_1b_n \neq 0$ since a_1 is a unit. That is, $a_1b_n = -$ the coefficient of x_n in $*$ which is then not zero. Now, since $*$ $\in R[[x^T]]$ we conclude that $n \in T$. This concludes our induction, hence we get $f^{-1} \in R[[x^T]]$. \square

REFERENCES

- [1] Doubilet P., Rota G.C., Stanley R. (1972). On the foundations of combinatorial theory. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability 2*: 267-318
- [2] Berstel J., Reutenauer C. (1988). Rational Series and Their Languages. *Monographs in Theoretical Computer Science, Springer-Verlag Berlin Heidelberg* Volume 12.
- [3] Dirichlet, P. G. L. Sur l'usage des séries infinies dans la théorie des nombres, *Journ. f. Math.* 18 (1838) 259-74.
- [4] Droste, M.; Zhang, G. On transformations of formal power series. *J. Inform. Comput.* **184** (2003), 369–383.
- [5] Hardy, G.H.; E.M. Wright An Introduction to the Theory of Numbers. *Oxford University Press* (1938).
- [6] Bousquet-Mélou, M. Rational and algebraic series in combinatorial enumeration. *Proc. Intern. Congr. Math.* (2006), 789–826.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY
E-mail address: e.enochs@uky.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY
E-mail address: jendaov@auburn.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY
E-mail address: fzo0005@auburn.edu